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Calculating the infra-red limit of quantum field theory using analyticity of correlation functions

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Abstract: We describe a general method for calculating the infra-red limit of physical quantities in unitary quantum field theories. Using analyticity of Green functions in a complex scale parameter, the infra-red limit is expressed as a contour integral entirely in the ultra-violet region. The infra-red limit is shown to be the limit of the Borel transform of the physical quantity. The method is illustrated by calculating the central charge of the perturbed unitary minimal models and the critical exponents of φ^4 theory in three dimensions. We obtain approximate values for the central charge which are very close to the exact values using only a one loop perturbative calculation. For φ^4 theory we obtain estimates which are within the errors of other more elaborate approaches.

Keywords: Renormalization Regularization and Renormalons, Field Theories in Lower Dimensions.

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1. Introduction and summary

It is important to get a better understanding of the infra-red limit of quantum field theory. Conventionally this is studied by extrapolating perturbation theory from the ultra-violet using the renormalisation group. The purpose of this paper is to show how to augment this approach with information derived from the analyticity properties of Green functions. These properties are well-known. For example, the Källen-Lehmann spectral representation of the two-point function of a scalar field shows that it has an analytic continuation to the complex momentum plane. We will use this to express the infra-red limit as an integral in the ultra-violet region where perturbation theory is applicable. For illustration assume that the physical quantity F depends on some distance scale s, and that $F(s)$ is analytic in the complex plane with the negative axis cut away, then the contour integral

$$\frac{1}{2\pi i}\int_C \frac{ds}{s} e^{\rho/s} F(s) = 0$$

vanishes, where the contour C is given in figure 1. This means that we can write the infra-red limit $F_{IR} = \lim_{|s|\to\infty} F(s)$ as an integral

$$F_{IR} = \frac{1}{2\pi i}\left(\int_{C_0} + \int_{C_1}\right)\frac{ds}{s}e^{\rho/s}F(s)$$

over the infinitesimal circle C_0 and the cut C_1.

We show below that the integral over the cut C_1 vanishes for large ρ, so that the infra-red limit F_{IR} then is given as an integral in the UV region over C_0 where perturbation theory applies. We call this integral $I(\rho)$ and get that $F_{IR} = \lim_{\rho\to\infty} I(\rho)$.

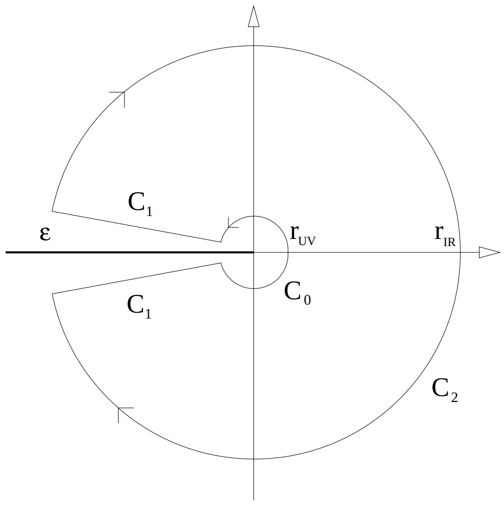

Figure 1: Integration contour C in the cut complex plane.

$I(\rho)$ is shown to be the Borel transform of $F(s)$ of order k, where k is related to the way we analytically continue in the complex scale parameter. The Borel transform has also been applied to perturbation theory in [26] where the Borel transform in the coupling is used. In this case assumptions (justified by details of the dynamics) have to be made about the poles of the Borel transform. The essential difference compared with our approach is that we transform with respect to the scale parameter. The analytic properties of Green functions, which are needed to define the Borel transform, in this case follow directly from general principles of quantum field theory.

We will use this method on two examples where the infra-red limit of the physical quantity $F(s)$ in question is well known, namely the central charge of the unitary minimal models perturbed by the relevant field $\phi_{(1,3)}$, and the critical exponents of φ^4 theory in three dimensions.

We also considered the perturbed minimal models in [1] where this method was shown to improve upon renormalisation group improved perturbation theory by a factor 2. $F(s)$ is given as the two point function of the energy momentum tensor, and we will here address some of the details left out of our discussion in [1]. We first show the analytic properties of $F(s)$, and we then obtain a bound on the exact non perturbative expression for $I(\rho)$, and this allows us to obtain an approximate value for calculating the central charge.

We employ this approximation method to calculate the central charge for the free fermionic and bosonic theories perturbed by a mass term and the unitary minimal models perturbed by the relevant operator $\phi_{(1,3)}$. The free massive theories can be solved exactly and these then demonstrate how $I(\rho)$ behaves in an exact case. For the perturbed minimal models we first calculate $F(s)$ as a renormalisation group improvement of a one loop perturbative calculation and show that this expression has the correct limit when $m \to \infty$, where m characterises the minimal model. We then show how our method improves upon the perturbative renormalisation group result. If we use the largest domain of analyticity for $F(s)$ apparent from the spectral decomposition there is something of a surprise. The approximation becomes dramatically close to the exact value, which is remarkable as it is only based on one loop perturbation theory.

The other example is φ^4 theory in three dimensions, where we calculate the critical exponents ν and η. φ^4 theory has been shown to be Borel summable of order one in the perturbative expansion in the coupling constant. The perturbative series will often at most be asymptotic, if for example a theory is ill defined for negative couplings the convergence radius will be zero. The question is then if the perturbative series captures all of the physics when it is only asymptotic. This will depend on the analytic properties of the correlators, and if the theory is Borel summable perturbation theory will contain all information about the theory. We use the propagator to define functions F_ν, F_η which have the critical exponents ν and η as their infra-red limit. All scale dependence is moved into the coupling using the running coupling so that $F_\nu(s)$ and $F_\eta(s)$ are given as asymptotic series in the scale parameter s, and we show their analyticity. In this example the actual calculation of the physical quantities ν and η is different form the minimal models case, as we cannot use the same approximation. The reason is that in this case we use higher order perturbative expressions (5,6 and 7 loops) and one cannot find an exact expression for the perturbative running coupling (unlike the one loop case). The infra-red values for ν, η are therefore obtained from the limiting value of the Borel transform, and to calculate this we use a conformal mapping and Padé approximation which is done in the appendix. The estimates we obtain for ν, η are within the errors of other more elaborate approximations.

The main results are that we can use analyticity in the scale parameter to express the infra-red physics as an integral in the ultra-violet region, the infra-red limit is then obtained taking the limit in the Borel transformed quantity. We show how this works in two examples with well known infra-red limits.

The outline is as follows: In the next section we will discuss analyticity of correlators, introduce the contour integral giving the infra-red limit and show that the contribution from the cut vanishes for large ρ. We also show that the contour integral is the Borel transform of $F(s)$. In section 3 we define $F(s)$ for the central charge, show its analyticity and discuss an approximation method for calculating the central charge. An exact bound on $I(\rho)$ is given. In section 4 we use this approximation on the free massive bosonic and fermionic theory together with the minimal models perturbed by $\phi_{(1,3)}$. In section 5 we discuss the φ^4 theory and introduce a function giving the critical exponents ν, η in the infra-red limit, and obtain estimates of ν and η from the limit of the Borel transform.

2. Defining the contour integral

The physical quantities of a quantum field theory can be written in terms of Green functions which have well known analyticity properties. As an example consider the two point function of a scalar field $A(p)$. The Källen-Lehmann spectral representation writes this as a sum over intermediate states

$$\langle A(p)A(0)\rangle = \int_0^\infty d\mu^2\ \tilde{c}(\mu^2)\frac{1}{p^2+\mu^2}\,,$$

where $d\mu^2\tilde{c}(\mu^2)$ is the spectral density. Introducing the complex scale factor $s \in \mathbb{C}$

$$\langle A(sp)A(0)\rangle = \int_0^\infty d\mu^2\ \tilde{c}(\mu^2)\frac{1}{s^2p^2+\mu^2}$$

then shows that $\langle A(sp)A(0)\rangle$ is analytic in the positive half-plane $\mathrm{Re}(s) > 0$.

Generally physical quantities will not be analytic in the positive half-plane but in some sector $S(\alpha)$ of the complex plane, defined as

$$S(\alpha) = \left\{ z = re^{i\phi}\,|\, 0 < r < \infty,\ -\frac{\alpha}{2} < \phi < \frac{\alpha}{2} \right\}. \tag{2.1}$$

Note that $0 \notin S(\alpha)$. We want to calculate the infra-red limit F_{IR} of a physical quantity $f(x)$, where $F_{IR} = \lim_{|x|\to\infty} f(x)$, and we will denote the UV limit $F_{UV} = \lim_{|x|\to 0} f(x)$. We are considering functions which have well defined limits in $S(\alpha)$, i.e. $\lim_{|s|\to\infty} F(s) = F_{IR}$ and $\lim_{s\to 0} F(s) = F_{UV}$ for $s \in S(\alpha)$.

In the two examples we will consider the physical quantities $f(x)$ are given as functionals of a two point correlator resulting in analytic functions $F(s)$. A large class of physical quantities will in this way preserve the analytic structure of the Green functions in the theory.

The scale parameter s can of course be introduced into the physical quantity in a number of ways. We will define by $\tilde{F}(s)$ the analytical continuation of $f(x)$ where s is introduced as $\tilde{F}(s) \equiv f(sx)|_{x=1}$ so that $\tilde{F}(s)$ has the expansion $\tilde{F}(s) = \sum \tilde{F}_n s^n$ around the origin, and $\tilde{F}(s)$ is analytic in $S(\alpha)$. Choosing another positive power $f(s^\gamma x)|_{x=1}$, $\gamma > 0$, gives the same ultra-violet and infra-red limits, but different intermediate behaviour. Let us now introduce the scale parameter so that the opening of the analytic sector is $2(\pi - \epsilon')$ with $\epsilon' \ll 1$. $F(s) = \tilde{F}(s^a)$ is analytic in $S = S(2(\pi - \epsilon'))$ provided that $a = \alpha/2(\pi - \epsilon')$. We will now express the infra-red limit as an integral entirely in the ultra-violet region using analyticity.

Using Cauchy's theorem the analyticity of $F(s)$ implies that the contour integral

$$\frac{1}{2(\pi-\epsilon)i}\int_C ds\,\frac{e^{\rho/s}}{s}F(s) = \frac{1}{2(\pi-\epsilon)i}\left(\int_{C_0} ds + \int_{C_1} ds + \int_{C_2} ds\right)\frac{e^{\rho/s}}{s}F(s) = 0\,, \tag{2.2}$$

vanishes in the sector S, where $\rho \in \mathbb{R}_+$ and the contour C is given in figure 1; $\epsilon > \epsilon'$ so that $C \subset S$. The contribution from the contour C_2 in (2.2) (see figure 1) in the limit where $r_{IR} \to \infty$ becomes

$$\lim_{r_{IR}\to\infty} \frac{1}{2(\pi-\epsilon)} \int_{\pi-\epsilon}^{-\pi+\epsilon} d\theta\, e^{\rho e^{-i\theta}/r_{IR}} F(r_{IR}e^{i\theta}) = -\lim_{|s|\to\infty} F(s) = -F_{IR}\,. \tag{2.3}$$

The angular integral and the limit $r_{IR} \to \infty$ can be interchanged as the integrand in (2.2) is bounded by a constant in the limit $r_{IR} \to \infty$. We then get the integral representation of F_{IR}

$$F_{IR} = \frac{1}{2(\pi-\epsilon)i}\left(\int_{C_0} ds \frac{e^{\rho/s}}{s} F(s) + \int_{C_1} ds \frac{e^{\rho/s}}{s} F(s)\right). \tag{2.4}$$

Analogously by considering

$$\frac{1}{2(\pi-\epsilon)i}\int_C ds\,\frac{e^{\rho s}}{s} F(s) = \frac{1}{2(\pi-\epsilon)i}\left(\int_{C_0} ds + \int_{C_1} ds + \int_{C_2} ds\right)\frac{e^{\rho s}}{s} F(s) = 0\,, \tag{2.5}$$

we get that

$$F_{UV} = \frac{-1}{2(\pi-\epsilon)i}\left(\int_{C_2} ds \frac{e^{\rho s}}{s} F(s) + \int_{C_1} ds \frac{e^{\rho s}}{s} F(s)\right). \tag{2.6}$$

We denote the integral along the contour C_1 (close to the cut) by

$$\mathrm{cut}(\rho) = \frac{1}{2(\pi-\epsilon)i}\int_{C_1} ds \frac{e^{\rho/s}}{s} F(s)\,. \tag{2.7}$$

The integrand in this integral is damped by the factor $e^{-\rho/|s|}$, and we will show that $\lim_{\rho\to\infty}\mathrm{cut}(\rho) = 0$ by showing that it is bounded by a finite integral for all ρ allowing us to take the limit $\rho \to \infty$ in the integrand. We substitute $s = re^{\pm i(\pi-\epsilon)}$ for points on C_1 in the upper and lower half-plane, the integral becomes

$$\begin{aligned}
\mathrm{cut}(\rho) &= \frac{-1}{2(\pi-\epsilon)i}\left(\int_{r_{IR}}^{r_{UV}} dr \frac{e^{\rho e^{-i(\pi-\epsilon)}/r}}{r} F(re^{i(\pi-\epsilon)}) + \int_{r_{UV}}^{r_{IR}} dr \frac{e^{\rho e^{i(\pi-\epsilon)}/r}}{r} F(re^{-i(\pi-\epsilon)})\right) \\
&= \frac{-1}{(\pi-\epsilon)}\int_{r_{UV}}^{r_{IR}} dr \frac{e^{\rho\cos(\pi-\epsilon)/r}}{r}\, \mathrm{Im}[e^{i\rho\sin(\pi-\epsilon)/r} F(re^{-i(\pi-\epsilon)})] \\
&= \frac{-1}{(\pi-\epsilon)}\int_{r_{UV}}^{r_{IR}} dr \frac{e^{\rho\cos(\pi-\epsilon)/r}}{r}\left(\cos\left(\rho\sin\frac{(\pi-\epsilon)}{r}\right)\mathrm{Im}[F(re^{-i(\pi-\epsilon)})] + \right. \\
&\qquad\left. + \mathrm{Re}[F(re^{-i(\pi-\epsilon)})]\sin(\rho\sin(\pi-\epsilon))\right).
\end{aligned}$$

We divide the r interval into $(r_{UV}, 1)$ and $(1, r_{IR})$ and write $\mathrm{cut}(\rho) = \mathrm{cut}_{UV} + \mathrm{cut}_{IR}$. The function $F(s)$ is finite in S hence there exists a constant $q > 0$ so that in the limit $r_{UV} \to 0$

$$|\,\mathrm{cut}_{UV}\,| < q \int_0^1 dr \frac{e^{-\rho/r}}{r} \to 0\,, \qquad \text{for} \quad \rho \to \infty\,. \tag{2.8}$$

We know that $F(s) \to F_{IR} \in \mathbb{R}$ for $|s| \to \infty$ from S, hence $\mathrm{Im}[F(re^{-i(\pi-\epsilon)})] \to 0$ for $r \to \infty$. If $\mathrm{Im}[F]$ falls off like $r^{-\delta}$ for some $\delta > 0$ then in the limit $r_{IR} \to \infty$, $\exists\, k_1, k_2, k_3 > 0$:

$$|\mathrm{cut}_{IR}| < k_1 \int_1^\infty dr \frac{\left(\sin(k_2/r) + k_3 r^{-\delta}\right)}{r} < \infty\,, \qquad \text{for all } \rho \in \mathbb{R}_+\,. \tag{2.9}$$

For a general $F(s)$ where the fall off might be slower we keep a finite r_{IR} then $|\,\mathrm{cut}_{IR}\,|$ is again finite for all ρ and a finite r_{IR} introduces a $O(\frac{1}{r_{IR}})$ term in (2.4) which is negligible for r_{IR} large. As $\epsilon \ll 1$ we can replace $\frac{1}{2(\pi-\epsilon)i}$ with $\frac{1}{2\pi i}$ given a term $O(\epsilon)$ on the right hand side in (2.4) which is again negligible for small ϵ. We can then define

$$I_{IR}(\rho) = \lim_{r_{UV} \to 0} \frac{1}{2\pi i} \int_{C_0} ds \frac{e^{\rho/s}}{s} F(s) \tag{2.10}$$

so that $F_{IR} = \lim_{\rho\to\infty} I_{IR}(\rho)$ and we have succeeded in writing the infra-red limit as the limiting value of a contour integral in the ultra-violet region.

We will now use Cauchy's theorem again to rewrite $I_{IR}(\rho)$ and this will show that $I_{IR}(\rho)$ is the Borel transform of $\tilde{F}(s)$. Instead of integrating over C_0 we will integrate over $\tilde{C}$ given as the path from the origin along the ray $\arg(s) = -\pi + \epsilon$ and then anti-clockwise along $|s| = \tilde{x} > 0$ until $\arg(s) = \pi - \epsilon$ and then back to the origin. It follows that this contour integral is independent of the choice of $\tilde{x}$ and ϵ as long as $\epsilon' < \epsilon < \pi/2$. The upper limit ensures that the rays stay in the negative half-plane (where $e^{\rho/s}$ is a damping factor), and the lower limit that $F(s)$ is analytic on the contour. Using this contour the ultra-violet limit $r_{UV} \to 0$ can be taken explicitly by extending the rays to the origin.

Using that $F(s) = \tilde{F}(s^a)$ for $a = \alpha/2(\pi - \epsilon')$ then amounts to

$$I_{IR}(\rho) = \frac{1}{2\pi i} \int_{\tilde{C}} \frac{ds}{s} e^{\rho/s} F(s) = \frac{1}{2\pi i} \int_{\tilde{C}} \frac{ds}{s} e^{\rho/s} \tilde{F}(s^a) = \frac{k}{2\pi i} \int_{\tilde{C}'} \frac{ds}{s} e^{(\frac{\tilde{\rho}}{s})^k} \tilde{F}(s)\,, \tag{2.11}$$

where $k = 1/a$, $\tilde{\rho} = \rho^{1/k}$ and $\tilde{C}'$ is the contour where the rays satisfy $|\arg(s)| = |(\pi - \epsilon'')/k|$ for any $\epsilon'' \in (\epsilon', \pi/2)$. We will write $\epsilon'' = (\pi - \tilde{\epsilon})/2$ for any $\tilde{\epsilon} \in (0, \pi - 2\epsilon')$, then $|\arg((\tilde{\rho}/s)^k)| > \pi/2$ and the integrand in (2.11) is again damped on the rays. Also, on the rays is $|\arg(s)| = (\pi + \tilde{\epsilon})/2k = \frac{\alpha}{4}\frac{\pi+\tilde{\epsilon}}{\pi-\epsilon'} < \alpha/2$ so that $\tilde{F}(s)$ is analytic on the contour, $\tilde{C}' \subset S(\alpha)$, and the contour integral is therefore well defined.

Equation (2.11) then shows that $I_{IR}(\tilde{\rho})$ is the Borel transform of $\tilde{F}(s)$ of order k [16]. The Borel transform of order k of a formal power series $h(z) = \sum_n h_n z^n$ is given by $\mathcal{B}_k(h)(\rho) = \sum_n \frac{h_n \rho^n}{\Gamma(1+n/k)}$, using the integral representation $1/\Gamma(1+n) = \frac{1}{2\pi i}\int_{\tilde{C}'} \frac{ds}{s} e^{1/s} s^n$ it then follows that (2.11) is the Borel transform of order k of $\tilde{F}(s)$ by inserting a series expansion for $\tilde{F}(s)$.

The contour integral in (2.11) is independent of the contour $\tilde{C}'$, i.e. in the choice of $\tilde{\epsilon}$ for $\tilde{\epsilon} \in (0, \pi - 2\epsilon')$. This shows that equation (2.11) holds for all k with $F(s) \equiv \tilde{F}(s^{1/k})$ and $\alpha > \pi/k$, because we can always find an $\tilde{\epsilon} > 0$ so that $\alpha > (\pi + \tilde{\epsilon})/k$, which again implies that $\tilde{C}' \subset S(\alpha)$ and $|\arg(\tilde{\rho}/s)^k| > \pi/2$ which again makes the contour integral well defined.

We have then shown that: *the infra-red limit of a physical quantity, F_{IR}, is the limiting value of the Borel transform of $\tilde{F}(s)$, and changing the way in which the scale parameter is introduced amounts to changing the order of the Borel transform. The order k has to satisfy the bound $\alpha > \pi/k$ where α determines the analytic sector of the physical quantity $\tilde{F}(s)$.*

One way of calculating the contour integral in (2.11) is to insert a series expansion of $\tilde{F}(s)$ around the origin, but because $\tilde{F}(s)$ is only analytic in a sector this series can only be an asymptotic expansion of $\tilde{F}(s)$.[1] An analytic function has a unique asymptotic expansion, but an asymptotic series might be the asymptotic expansion of several analytic functions. However in the case where a function is analytic in a sector of opening $\alpha > \pi/k$, which is the situation we have above, the map between the analytic function and the asymptotic series of order k is injective (but not surjective) [16]. This means that all information about the exact function is contained in the asymptotic series.

An asymptotic series of $\tilde{F}(s)$ in the scale parameter can be obtained doing the perturbative expansion in the coupling and then inserting the running coupling constant. Assume that $\tilde{F}$ is given as a formal perturbative series, which we write as $\tilde{F}(s, g) = \sum_n \hat{F}_n(s) g^n$. The Callan-Symanzik equation states that a theory is invariant, i.e. the correlators are invariant, under a scale transformation if the couplings change according to the renormalisation group. A scaled quantum field theory can therefore equivalently be described by a theory on the same scale, but with couplings changed according to the renormalisation group. In this way scale dependence of a theory can be moved into the running coupling $\bar{g}(s)$. Moving all scale dependence into the running coupling we get an asymptotic series in the scale $\tilde{F}(s = 1, \bar{g}(s)) = \sum \tilde{F}_n s^n$.[2]

[1] If $f(z)$ is an analytic function in a neighbourhood of the origin it follows that its asymptotic series around the origin is convergent and equal to the power series of $f(z)$; the lack of analyticity at the origin forces the asymptotic series to diverge. A formal power series $f(z) = \sum f_n z^n$ is asymptotic of order k if $\exists\, C, K > 0 : |f_n| < C K^n \Gamma(1 + n/k)$, this shows that the Borel transform of order k, of such a series, has a non zero convergence radius.

[2] If $\tilde{F}(s)$ is asymptotic of order k' ($\tilde{F}(s) \in \mathbb{C}[[s]]_{1/k'}$) we will choose $k = k'$, and then the Borel

The examples we consider below have $\alpha = \pi - \epsilon'$, where $\epsilon' \ll 1$, hence from the constraint $\alpha > \pi/k$: $k = 1 + \delta$ for some $\delta > 0$. We will generally try to minimise the order k ($k \to 1$), thus maximising the analytic sector and the convergence of the Borel transform.

3. Defining $\tilde{F}(s)$ for the central charge

As the first example we want to calculate the infra-red central charge of a 2 dimensional unitary quantum field theory. The renormalisation group flow of the central charge is governed by Zamolodchikov's c-theorem [4]. This states that for a unitary and renormalisable quantum field theory in 2 dimensions there exists a function which is monotonically decreasing along the renormalisation group flow, and which is stationary only for conformally invariant theories where it takes the value of the Virasoro central charge. The c-theorem implies that the infra-red limit, where the scale goes to infinity, and the ultra-violet limit, where the scale vanishes, are fixed points of the renormalisation group. In 2 dimensions scale invariance implies conformal invariance so in these scaling limits we have a conformal field theory characterised by the central charge. We define the energy-momentum tensor as in [2]

$$\langle T_{\mu\nu}(x)\rangle = \frac{2V}{\sqrt{g(x)}}\frac{\delta W[g]}{\delta g^{\mu\nu}}\,, \tag{3.1}$$

where $V = \mathrm{Vol}(S^{n-1}) = 2\pi$ in 2D and $W[g]$ is the effective action.

The correlator $\langle T_{zz}(z)T_{zz}(0)\rangle$ gives the ultra-violet and infra-red central charges in the limits $z \to 0$ and $z \to \infty$ respectively. In [2] $\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle$ was written using the Källen-Lehmann spectral representation:

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{\pi}{3\cdot 16}\int_0^\infty d\mu^2\,\tilde{c}(\mu^2)\int \frac{d^2p}{(2\pi)^2}e^{ipx}\,\frac{(g_{\mu\nu}p^2 - p_\mu p_\nu)(g_{\rho\sigma}p^2 - p_\rho p_\sigma)}{p^2+\mu^2}\,, \tag{3.2}$$

and it follows that

$$\langle T_{zz}(z,\bar{z})T_{zz}(0,0)\rangle = \frac{\pi}{3\cdot 16}\int_0^\infty d\mu^2\,\tilde{c}(\mu^2)\int \frac{d^2p}{(2\pi)^2}\frac{e^{\frac{i}{2}(p\bar{z}+\bar{p}z)}}{p\bar{p}+\mu^2}\,\bar{p}^4\,, \tag{3.3}$$

where we use the usual complex variables[3] z, $\bar{z}$ and $\tilde{c}(\mu^2)d\mu^2$ is the spectral density which represents the density in degrees of freedom of the quantum field theory at the mass μ. If we scale $z, \bar{z}$ by a positive real dimensionless parameter s ($s > 0$) we get

$$\langle T_{zz}(sz,s\bar{z})T_{zz}(0,0)\rangle = \frac{\pi}{3\cdot 16}\int_0^\infty d\mu^2\,\tilde{c}(\mu^2)\int \frac{d^2q}{(2\pi)^2}\frac{e^{\frac{i}{2}(q\bar{z}+\bar{q}z)}}{q\bar{q}+s^2\mu^2}\,\frac{\bar{q}^4}{s^4}\,. \tag{3.4}$$

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transform of $\tilde{F}$ has a non-zero convergence radius and the analytically continued value at infinity uniquely determines F_{IR}. The analytical continuation can be done by a conformal mapping, we will discuss this in the appendix.

[3] $z = t_E + ix$, $\bar{z} = t_E - ix$ and $d^2z \equiv d^2x = dx \wedge dt_E = -\frac{i}{2}dz\wedge d\bar{z}$.

In the ultra-violet limit where $s \to 0$ then $\frac{\bar{q}^4}{q\bar{q}+s^2\mu^2} \to \frac{\bar{q}^4}{q\bar{q}}$ and (3.4) becomes[4]

$$\langle T_{zz}(sz, s\bar{z})T_{zz}(0,0)\rangle \to \frac{1}{2s^4z^4}\int_0^\infty d\mu^2\,\tilde{c}(\mu^2) = \frac{c_{UV}}{2s^4z^4}\,, \qquad \text{for } s\to 0\,. \tag{3.5}$$

The ultra-violet central charge is therefore $c_{UV} = \int_0^\infty d\mu\,\tilde{c}(\mu)$. To calculate the infra-red limit we first note that

$$\int \frac{d^2q}{(2\pi)^2}\frac{e^{\frac{i}{2}(q\bar{z}+\bar{q}z)}}{q\bar{q}+s^2\mu^2}\bar{q}^4 = 2^4\left(\frac{\partial}{\partial z}\right)^4 G(z,\bar{z},s\mu)\,, \tag{3.6}$$

where $G(z,\bar{z},\mu)$ is equal to the free Bose propagator at mass μ. $G(z,\bar{z},\mu)$ can be written in terms of a modified Bessel function [5]

$$G(x,\mu) = \int \frac{d^2p}{(2\pi)^2}\frac{e^{ix\cdot p}}{p^2+\mu^2} = \frac{K_0(|x|\mu)}{2\pi}\,. \tag{3.7}$$

Performing the differentiation we can then write (3.4) as

$$\begin{aligned}\langle T_{zz}(sz, s\bar{z})T_{zz}(0,0)\rangle = \frac{1}{2\cdot 48\,s^4z^4}\int_0^\infty d\mu^2\;\tilde{c}(\mu^2)\mu s|z|((\mu^3s^3|z|^3 + \\ +24\mu s|z|)K_0(\mu s|z|) + (8\mu^2s^2|z|^2+48)K_1(\mu s|z|))\,.\end{aligned} \tag{3.8}$$

In the infra-red limit where $s\to\infty$, $K_0(\mu|z|s)$ and $K_1(\mu|z|s)$ have the asymptotic behaviour [6] $e^{-\mu s|z|}$ and the only contribution to (3.8) comes from the massless limit where $\mu\to 0$, hence

$$\langle T_{zz}(sz, s\bar{z})T_{zz}(0,0)\rangle \to \lim_{\epsilon\to 0}\frac{1}{2s^4z^4}\int_0^\epsilon d\mu^2\,\tilde{c}(\mu^2) = \frac{c_{IR}}{2s^4z^4}\,, \qquad \text{for } s\to\infty\,, \tag{3.9}$$

so that $c_{IR} = \lim_{\epsilon\to 0}\int_0^\epsilon d\mu^2\,\tilde{c}(\mu^2)$. This shows that $c_{UV} \geq c_{IR}$ as the spectral density is positive for a unitary theory, and this is another way of showing the c-theorem [2]. This representation of the central charge using the spectral representation also shows that the central charge measures the number of (massless) degrees of freedom of the CFT. We define the function[5]

$$\tilde{F}(s) = 2z^4s^4\langle T_{zz}(sz,s\bar{z})T_{zz}(0,0)\rangle\big|_{z=\bar{z}=1} \tag{3.10}$$

here s and $\tilde{F}(s)$ are dimensionless and $s\in\mathbb{R}_+$. This function then satisfies

$$\tilde{F}(s) \to \begin{cases} c_{UV} & \text{for } s\to 0_+\,, \\ c_{IR} & \text{for } s\to\infty\,. \end{cases} \tag{3.11}$$

[4] The Fourier transform of $(\pi/24)\bar{p}^4/\bar{p}p$ is $1/z^4$.

[5] When we set $|z|=1$ then μ becomes dimensionless in (3.8), we also denote this dimensionless quantity by μ.

3.1 Analyticity of $\tilde{F}(s)$

We will show that $\tilde{F}(s)$ is an analytic continuation of $\tilde{F}(x)$, $x \in \mathbb{R}_+$, for $s \in S = S(\pi - \epsilon')$ (with $\epsilon' \ll 1$); to show this write $\tilde{F}(s) = \int_M d\nu f(s,\nu)$. For $\tilde{F}$ to be holomorphic in S then $f(s,\nu)$ must be holomorphic in S for all $\nu \in M$, and both f and df/ds must be integrable over the set M. Using (3.8) above we can write $\tilde{F}(s)$ as

$$\tilde{F}(s) = \frac{1}{48}\int_0^\infty d\mu^2\, \tilde{c}(\mu^2)\mu s\left((\mu^3 s^3 + 24\mu s)K_0(\mu s) + (8\mu^2 s^2 + 48)K_1(\mu s)\right). \quad (3.12)$$

$K_\nu(z)$ is holomorphic in S so $f(s,\nu)$ is clearly holomorphic in S. The modified Bessel functions also satisfy that $|K_\nu(z)|$ is bounded for $|z| \geq \epsilon$ for any $\epsilon \in \mathbb{R}_+$ and $|\arg z| < \pi/2$, which is the case for $z = \mu s$ when $s \in S$. For large values of $|s|$ integrability is ensured by $\int_0^\infty d\mu^2\, \tilde{c}(\mu^2) = c_{UV} < \infty$ and the asymptotic behaviour $K_\nu(z) = \sqrt{\pi/2z}\, e^{-z}(1 + O(z^{-1}))$. Around the origin $K_0(z)z \to 0$ for $z \to 0$ and $K_1(z)z \to 1$ for $z \to 0$, hence $\tilde{F}(s)$ is integrable. $d\tilde{F}/(s)ds$ is shown to be integrable in a similar way and the analyticity is shown.

From the form of $\tilde{F}(s)$ in (3.12) it follows that $\tilde{F}(s)$ has a limit value $\tilde{F}(s) \to d$ for $s \to 0$ from S. From (3.11) it follows that $d = c_{UV}$, and we define $\tilde{F}(0) = c_{UV}$. It also follows from (3.12) that $\tilde{F}(s)$ has a limit value for $|s| \to \infty$ from S: $\tilde{F}(s) \to d'$ and (3.11) again sets $d' = c_{IR}$.

3.2 The approximation for c_{IR}

We have the following representation of c_{IR} and c_{UV}

$$c_{IR} = \frac{1}{2\pi i}\left(\int_{C_0} ds \frac{e^{\rho/s}}{s} F(s) + \int_{C_1} ds \frac{e^{\rho/s}}{s} F(s)\right) = I_{IR}(\rho) + \mathrm{cut}(\rho) \quad (3.13)$$

$$c_{UV} = \frac{-1}{2\pi i}\left(\int_{C_2} ds \frac{e^{\rho s}}{s} F(s) + \int_{C_1} ds \frac{e^{\rho s}}{s} F(s)\right) = I_{UV}(\rho) + \mathrm{cut}(\rho)\,, \quad (3.14)$$

where in both cases the contribution from the cut is rapidly decreasing in ρ. Note that in these relations the integral is performed in the opposite scaling limit of the quantity we calculate. We will here concentrate on (3.13) as perturbation theory can be applied to $F(s)$ when $s \in C_0$, we will call $I_{IR}(\rho) = I(\rho)$.

We can choose coordinates in the coupling constant space so that the ultra-violet fixed point corresponds to $\bar{g}(s) = 0$. In the ultra-violet limit where $s \to 0$ we may describe $F(s)$ by perturbation theory as $\bar{g}(s) \to 0$. The nth order perturbative approximation of $F(s)$ is denoted by $F_n(s)$ and the corresponding integral by $I_n(\rho)$. In the limit of large ρ the contribution from the cut vanishes and we get

$$c_{IR} \approx \lim_{\rho\to\infty} I_n(\rho) = \lim_{s\to\infty} F_n(s)\,, \quad (3.15)$$

where the last equality follows setting $s' = s/\rho$ in (3.13) and then taking the limit $\rho \to \infty$ in the integrand valid for all $r_{UV} > 0$. Moving all scale dependence into the running coupling $\bar{g}(s)$ we can write $F_n(s) = \Phi_n(\bar{g}(s))$. Let g^*_{IR} denote the first non trivial zero of the perturbative $\beta-$function, i.e. $\bar{g}(s) \to g^*_{IR}$ for $s \to \infty$. Then equation (3.15) becomes

$$\lim_{s\to\infty} F_n(s) = \lim_{s\to\infty} \Phi_n(\bar{g}(s)) = \Phi_n(g^*_{IR}) = c^*_{IR}\,, \tag{3.16}$$

which is the perturbative estimate of c_{IR} we want to improve.

$F_n(s)$ is the ultra-violet perturbative approximation to $F(s)$ and the integration range in $I(\rho)$ is compact so $\lim_{\rho\to 0} I_n(\rho) = \lim_{\rho\to 0} I(\rho) = c_{UV}$. $I_n(\rho)$ therefore provides a good approximation to $I(\rho)$, for small enough ρ, since the power series expansion of $I(\rho)$ is controlled by the small s expansion of $F(s)$ for which perturbation theory applies. This is illustrated in (4.34) below for the minimal models. For larger values of ρ higher order terms in the expansion of $I_n(\rho)$ become important and the coefficients of the expansion of $I_n(\rho)$ and $F_n(\rho)$ part company. If $c_{IR} < c^*_{IR}$ and if the region where $I_n(\rho)$ is a good approximation to $I(\rho)$ is large enough, then $I_n(\rho)$ will have a minimum before approaching its limiting value of c^*_{IR}. Since this minimum occurs at the largest value of ρ for which $I_n(\rho)$ is a reasonable approximation to $I(\rho)$ and the true value of c_{IR} is given by $I(\infty)$, it is this minimum of $I_n(\rho)$ that we will use to provide a better estimate of c_{IR}. The approximation then becomes

$$c_{IR} = I_n(\rho_m)\,, \tag{3.17}$$

where ρ_m is the value where $I_n(\rho)$ attains its minimum. Below we consider the quantity $\Delta c_{exact} = c_{UV} - c_{IR}$ and denote the approximation to it $\Delta c_{approx} = c_{UV} - I_n(\rho_m)$, we call the perturbative value $\Delta c_{pert} = c_{UV} - \lim_{\rho\to\infty} I_n(\rho) = c_{UV} - c^*_{IR}$. This approximation rests upon the assumption that the exact function is monotonically decreasing from c_{UV} to c_{IR}, or at least that its minimum value is close to c_{IR}. Below we will show that this is indeed the case for $k = 4$, at least to a very good approximation.

3.3 Exact bound on Δc

As discussed above can the exact value $I(\rho)$ possibly be smaller than the asymptotic value c_{IR}. We will denote by ρ'_m the value where $I(\rho)$ attains its minimum and $\Delta c_{est} = c_{UV} - I(\rho'_m)$, then $\Delta c_{exact} - \Delta c_{est}$ measures the undershoot of the exact function $I(\rho)$ compared with its asymptotic value c_{IR} (see figure 3). From the spectral representation of $F(s)$ in (3.12) we can obtain a rigorous lower bound on this exact undershoot. In [2] it was shown that the spectral density can be written as

$$\tilde{c}(\mu^2) = c_{IR}\delta(\mu^2) + \hat{c}(\mu^2)\,,$$

we showed in (3.5) that $\int d\mu^2 \tilde{c}(\mu^2) = c_{UV}$ hence $\int d\mu^2 \hat{c}(\mu^2) = \Delta c_{exact}$. Using this in

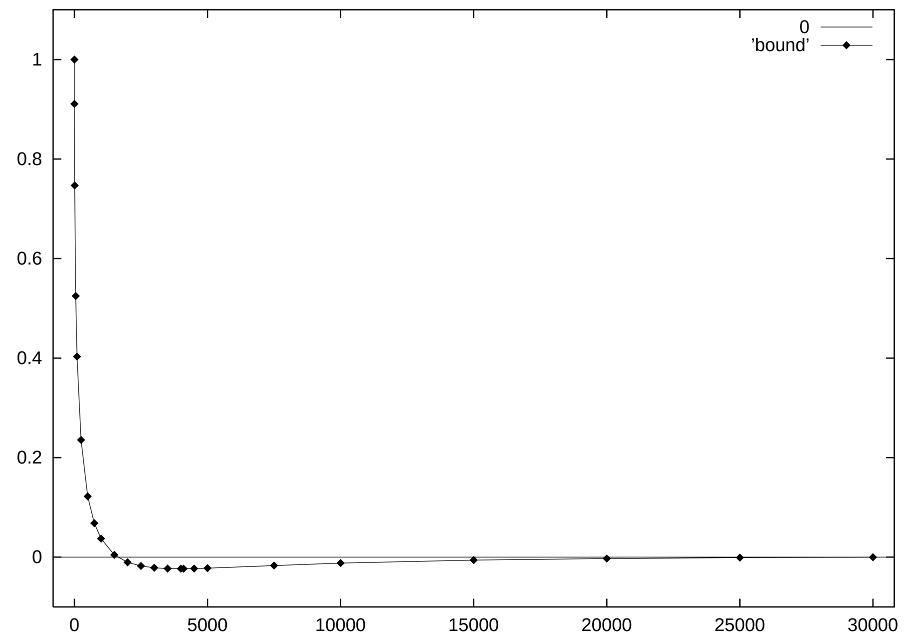

Figure 2: Numerical integration of $\Upsilon(\rho)$ given in (3.18).

(3.12) we get

$$\begin{aligned}\Delta c_{est} &= c_{UV} - I(\rho'_m) = c_{UV} - \frac{1}{2\pi i}\int_{\tilde{C}} \frac{ds}{s} e^{\rho'_m/s} F(s) \\ &= \Delta c_{exact} - \int d\mu^2\ \hat{c}(\mu^2)\Upsilon(\mu^2,\rho'_m)\,,\end{aligned}$$

where

$$\begin{aligned}\Upsilon(\mu^2,\rho'_m) = \frac{1}{2\pi i}\int_{\tilde{C}} \frac{ds}{s} e^{\rho'_m/s} \frac{\mu s^{1/4}}{48} \Big(\left((\mu s^{1/4})^3 + 24\mu s^{1/4}\right) K_0(\mu s^{1/4}) + \\ + (8(\mu s^{1/4})^2 + 48) K_1(\mu s^{1/4})\Big)\end{aligned} \tag{3.18}$$

and $F(s) = \tilde{F}(s^{1/k})$ with $k = 4$. Rescaling s allows us to move all μ dependence into $\rho'_m(\mu) = \rho'_m\mu^4$, so that we can write $\Upsilon(\mu^2,\rho'_m) = \Upsilon(\rho'_m(\mu))$. Unitarity ensures that $\hat{c}(\mu^2) \geq 0$ hence

$$\Delta c_{\text{exact}} - \Delta c_{est} \geq \min_{\mu} \Upsilon(\rho'_m(\mu)) \int_0^\infty d\mu^2 \hat{c}(\mu^2)\,, \tag{3.19}$$

$\Delta c_{est} > 0$ so that

$$0 \geqslant \frac{\Delta c_{\text{exact}} - \Delta c_{est}}{\Delta c_{\text{exact}}} \geqslant \min_{\rho>0} \Upsilon(\rho)\,. \tag{3.20}$$

The lower bound in the relative undershoot of $I(\rho'_m)$ therefore equals $\min_\rho \Upsilon(\rho)$. In figure 2 we plot $\Upsilon(\rho)$ for $k = 4$. With this value we get that $\min_\rho \Upsilon(\rho) = -0.0232$ so that the relative overshoot in Δc_{est} compared with Δc_{exact} is maximally 2.3%. It follows from (3.19) that the bound in (3.20) is only saturated for free theories (where $\hat{c}(\mu^2) \propto \delta(\mu^2 - m^2)$) for general interacting theories will the relative overshoot

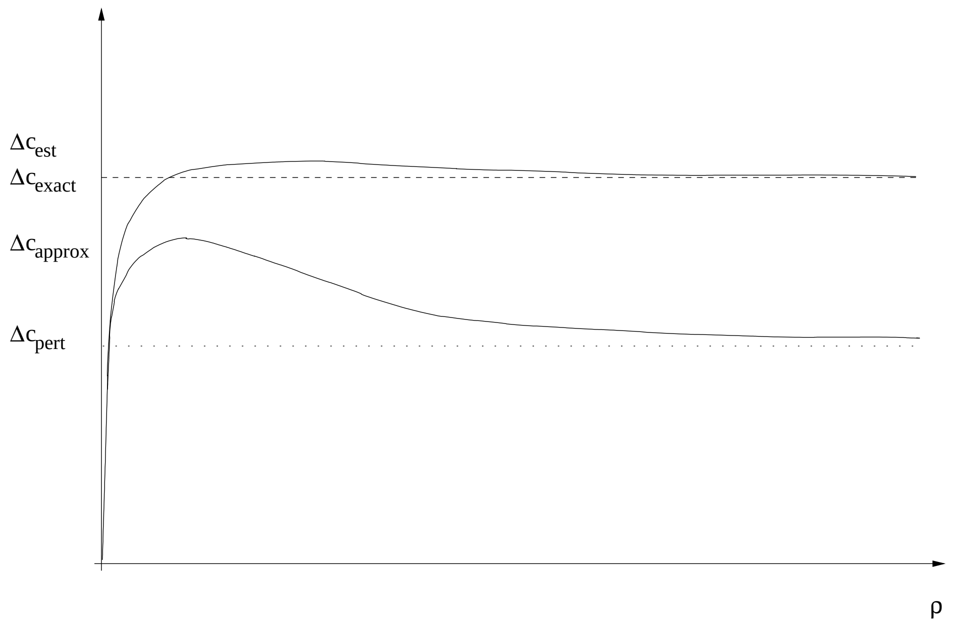

Figure 3: The expected behaviour of $c_{UV} - I_n(\rho)$ and $c_{UV} - I(\rho)$.

be smaller than $|\min_\rho \Upsilon(\rho)|$. We show in figure 3 the type of behaviour that we expect and which is confirmed for the minimal models below.

The actual choice of $k = 4$ then is a compromise between maximising the range of analyticity (small k) and minimising the undershoot in $I({\rho'}_m)$ (large k).

4. Application to different 2D models

In this section we calculate the central charge for the free bosonic and fermionic theory perturbed by a mass term. These theories are gaussian and we can calculate $F(s)$ exactly, the central charge can then be written as $\lim_{\rho\to\infty} I(\rho)$ as the contribution from the cut then vanishes. The infra-red limit of these theories is trivial ($c_{IR} = 0$) as all the degrees of freedom are massive, and therefore decouple when approaching the infra-red fixed point where the scale goes to infinity. The free theories are none the less important to consider as we can here obtain the exact function $I(\rho)$.

In section 4.3 we consider the unitary minimal models perturbed by the relevant operator $\phi_{(1,3)}$, this theory has a non-trivial infra-red fixed point. We obtain the renormalisation group improved perturbative calculation of c_{IR} and compare with the approximation (3.17).

4.1 The free boson

We take the action for the free bosonic theory in 2 dimensions with a mass m to be

$$S = \int d^2x \left(\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) + \frac{1}{2}m^2\varphi^2(x) \right). \tag{4.1}$$

The perturbation away from the conformal field theory is thus given by the purely massive term $\frac{1}{2}m^2\varphi^2$. The theory is still a free theory off criticality and the correlator $\langle TT\rangle$ can be calculated exactly in the whole scaling region from the ultra-violet to the infra-red. With this normalisation the energy-momentum tensor becomes

$$T(z,\bar z) = T_{zz}(z,\bar z) = -2\pi : \partial\varphi(z,\bar z)\partial\varphi(z,\bar z) : , \tag{4.2}$$

with the correlator

$$\begin{aligned}\langle T(z,\bar z)T(w,\bar w)\rangle &= (2\pi)^2\langle : \partial\varphi(z,\bar z)\partial\varphi(z,\bar z) :: \partial\varphi(w,\bar w)\partial\varphi(w,\bar w) :\rangle \\ &= 2(2\pi)^2\langle \partial\varphi(z,\bar z)\partial\varphi(w,\bar w)\rangle^2 ,\end{aligned} \tag{4.3}$$

as only the double contractions survive. Using the form (3.7) of the free propagator then (4.3) is $2(\partial_z\partial_z K_0(m|z|))^2$ where we have set $w=0$ using translation invariance. We now use the identities $K_n' = -\frac{1}{2}(K_{n-1}+K_{n+1})$, $K_2(x) = K_0(x) + \frac{2}{x}K_1(x)$, $K_{-1}(x) = K_1(x)$ and (4.3) becomes

$$\langle T(z,\bar z)T(0,0)\rangle = \frac{m^2|z|^2}{8z^4}\left(4K_1^2(m|z|)+m^2|z|^2K_0^2(m|z|)+4m|z|K_1(m|z|)K_0(m|z|)\right) , \tag{4.4}$$

hence $F(s)$ with $k=4$ becomes

$$F(s) = \left(\frac{ms^{1/2}}{2}\right)^2\left(\frac{4}{s^{1/2}}K_1^2(ms^{1/4}) + m^2K_0^2(ms^{1/4}) + \frac{4m}{s^{1/4}}K_0(ms^{1/4})K_1(ms^{1/4})\right) . \tag{4.5}$$

Knowing $F(s)$ exactly the central charges follow from (3.11) directly, but let us use the relations (3.14) and (3.13) in the limit of large ρ where the cut vanishes. For c_{UV} we substitute $s' = ms^{1/4}$, $\rho' = \rho/m^4$ in the contour C_2. From (3.14) we then get that (for $r_{IR} = 1/m^4$)

$$I_{UV}(\rho) = \frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta\, e^{4i\theta}e^{\rho' e^{2i\theta}}\left(4e^{-2i\theta}K_1^2(e^{i\theta}) + K_0^2(e^{i\theta}) + 4e^{-i\theta}K_0(e^{i\theta})K_1(e^{i\theta})\right) . \tag{4.6}$$

This is not expressible in terms of elementary functions, but it can be calculated using the analytical properties of the Bessel functions. The integration contour can be collapsed into a contour running along the imaginary axis together with an infinitesimal semi-circle around the origin. Setting $ix = e^{i\theta}$ we then get the 3 contributions

$$\frac{1}{4\pi i}\left(\int_{-1}^{-\epsilon}dx + \int_{\epsilon}^{1}dx + \int_{C_\epsilon}dx\right) e^{-\rho x^2}\left(-4xK_1(ix)^2 + x^3K_0(ix)^2 - 4ix^2K_0(ix)K_1(ix)\right) .$$

For $\rho\to\infty$ only the contribution from the infinitesimal semi-circle C_ϵ survive and here we can insert the asymptotic forms for $K_1(z) \sim z^{-1}$ and $K_0(z) \sim -\log z$.

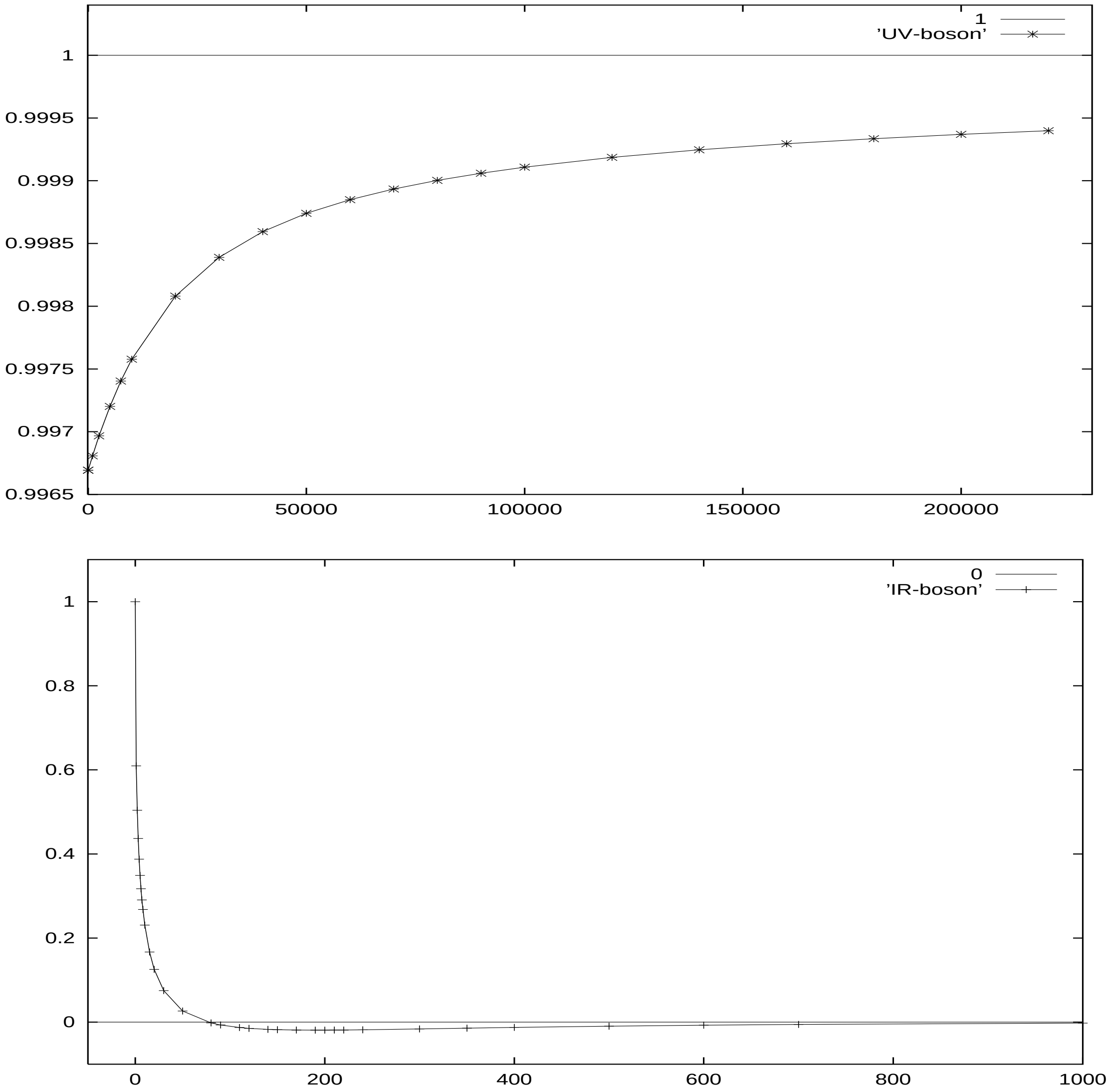

Figure 4: (a) Numerical integration of $I_{UV}(\rho)$ given by (4.6) compared with the exact value $c_{UV} = 1$. (b) Numerical integration of $I_{IR}(\rho)$.

In this limit only the first term with K_1^2 will contribute as is seen from the asymptotic form. Taking into account that we only integrate over half a circle we then get the well known result $c_{UV} = 1$, which is an exact result as the contribution from the cut vanishes in this limit.[6]

To calculate the infra-red central charge we use (3.15) hence $c_{IR} = \lim_{\rho\to\infty} I_{IR}(\rho) = \lim_{s\to\infty} F(s)$. From the asymptotic form of the modified Bessel functions $K_n(x) \sim e^{-x}$, together with (4.5), it follows that $c_{IR} = 0$. Figure 4a shows the exact function $I_{UV}(\rho)$ computed by a numerical integration of (4.6) using a NAG Fortran Library

[6]We could also directly have used the analogous of (3.15) namely $\lim_{\rho\to\infty} I_{UV}(\rho) = \lim_{s\to 0} F(s)$ which is again seen changing variable in (3.14) and taking the limit $\rho \to \infty$ in the integrand valid for all $r_{IR} < \infty$.

integration routine. The figure shows how the contribution from the cut vanishes in the limit of large ρ. The exact infra-red function $I_{IR}(\rho)$ is plotted in figure 4b, it is important to note that the minimum value of $I_{IR}(\rho)$ is very small, namely -0.019.

4.2 The free fermion

The free massive fermion has the action

$$S = \int d^2x \, \left(\bar{\psi}\partial\bar{\psi} + \psi\bar{\partial}\psi + im\bar{\psi}\psi\right). \tag{4.7}$$

A calculation analogous to the Bose case gives, for $k = 4$, that

$$F(s) = \frac{m^4 s}{8}\left(K_1^2(ms^{1/4})\left(1 + \frac{4}{m^2 s^{1/2}}\right) - K_0^2(ms^{1/4})\right). \tag{4.8}$$

The same arguments as above yields $c_{UV} = \frac{1}{2}$, and the infra-red contribution $c_{IR} = \lim_{s\to\infty} F(s) = 0$, again by using the asymptotic form of the Bessel functions. Again we can obtain the exact functions $I(\rho)$ by a numerical computation. We get the same behaviour of $I_{IR}(\rho)$ as in the bosonic case, but with the minimum value -0.0048, so this exact function does again not differ much from the infra-red central charge at its minimum value.

4.3 The unitary minimal models

The off critical quantum field theory picks out a specific renormalisation group flow from the ultra-violet to the infra-red conformal field theory. If the quantum field theory is in the neighbourhood of one of the renormalisation group fixed points λ^* in coupling constant space (we may be choose the coordinates so $\lambda^* = 0$ corresponds to the ultra-violet CFT) the action can be written as

$$S = S_{CFT} + \sum_{i=1}^{N} \lambda^i \int d^2x \, \Phi_i(x). \tag{4.9}$$

Here the $\Phi_i(x)$'s are scaling fields with scaling dimension Δ_i, the coupling constants λ^i then have mass dimension $[\lambda^i] = 2 - \Delta_i = y_i$. From (4.9) it follows that y_i is the renormalisation group eigenvalue of the scaling fields Φ_i. For renormalisable quantum field theories we need $y_i \geq 0$ and the scaling operators therefore have to be relevant ($y > 0$) or marginal ($y = 0$). For relevant operators we will move away from the fixed point when the scale increases and S_{CFT} thus corresponds to an ultra-violet critical point.

We consider quantum field theories which have the unitary minimal models $\mathcal{M}(m)$ as their scaling limits. The central charge for $\mathcal{M}(m)$ is given by

$$c(m) = 1 - \frac{6}{m(m+1)}, \qquad m = 3, 4, \ldots \tag{4.10}$$

There are $m(m-1)/2$ primary fields

$$\phi_{(p,q)} = \phi_{(m-p,m-q+1)}\,, \qquad 1 \le p \le m-1\,, \qquad 1 \le q \le m\,,$$

with the conformal dimensions

$$h_{(p,q)} = \bar{h}_{(p,q)} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}\,, \tag{4.11}$$

so the primary fields are scalars (spin zero). It follows from (4.11) that $2(m-2)$ of the primary fields satisfy $\Delta = 2h < 2$ and are thus relevant operators.[7] We will consider the minimal models perturbed by the relevant operator $\phi_{(1,3)}$ given by the action

$$S = S_{\mathcal{M}(m)} - \lambda_0 \int d^2x\ \phi_{(1,3)}(x)\,. \tag{4.12}$$

The reasons for choosing this operator are that:[8]

i) $\phi_{(1,3)}$ is a relevant field, $h_{(1,3)} = 1 - \frac{2}{m+1} < 1$, which exists in all $\mathcal{M}(m)$.

ii) $\phi_{(m,n)}$ form an algebra under the operator product expansion, from the fusion rules of minimal models it follows that the set $\phi_{(1,n)}$ constitutes a sub-algebra in which only $\phi_{(1,1)}$, $\phi_{(1,2)}$ and $\phi_{(1,3)}$ are relevant as seen from (4.11). $\phi_{(1,3)}$ is normalised so that the structure constant $C_{(1,3)(1,3)(1,1)} = 1$ where $I = \phi_{(1,1)} = \phi_{(m-1,m)}$ is the identity, and it has the self coupling $C_{(1,3)(1,3)(1,3)} = b(m)$. It does not couple to $\phi_{(1,2)}$ so it has no coupling to other relevant operators in the sub-algebra $\phi_{(1,n)}$ [8, 9, 13]. This means that there is a renormalisation group flow connecting the ultra-violet and infra-red fixed points along the direction of $\phi_{(1,3)}$ so it is consistent to include only the one relevant field $\phi_{(1,3)}$,[9] i.e. it is a geodesic renormalisation group trajectory [7].

iii) $\phi_{(1,3)}$ is the least relevant field, and the perturbation in (4.12) becomes marginal in the limit of $m \to \infty$ as $y = \frac{4}{m+1}$ so that $y \to 0$. In this limit the fixed points are arbitrarily close in coupling constant space, and perturbation theory is viable in the whole region from the ultra-violet to the infra-red.

[7] As the secondary fields will have a conformal dimension of at least $h+1$ there will be no relevant secondary fields.

[8] The model (4.12) is integrable and was first studied in [10]. The only other integrable perturbations of the unitary minimal models are with $\phi_{(1,2)}$ and $\phi_{(2,1)}$. These three models correspond respectively to the Korteweg-de Vries, Gibbon-Samede-Kotera and Kupersmidth equations [11].

[9] This can be seen by writing the Zamolodchikov metric in normal coordinates around the ultra-violet fixed point $g = 0$, $G_{ij} = \delta_{ij} + O(g^2)$, the beta-functions become in these coordinates $\beta^i(g) = -y^i g_i - \pi \sum_{j,k} C^i_{jk} g^j g^k + O(g^3)$. If $g^i \neq 0$ only for $g^{(1,3)}$ and $C_{(1,3)(1,3)j} = 0$ for $j \neq (1,3)$ then $\beta^j(g^{(1,3)}) = 0$ for $j \neq (1,3)$ and there is no flow transverse to the $\phi_{(1,3)}$ direction [3].

We want to calculate the difference between the ultra-violet and the infra-red central charge $\Delta c = c_{UV} - c_{IR}$. It has been argued that the infra-red conformal field theory of (4.12) is given by the unitary minimal model $\mathcal{M}(m-1)$, as Δc in the perturbative limit $y \to 0$ is given by $\Delta c = \frac{3}{16}y^3 + O(y^4)$ [8, 9], and from (4.10) we get that

$$c(m) - c(m-1) = \frac{12}{m(m^2-1)} = \frac{3y^3}{2(2-y)(4-y)} = \frac{3y^3}{16} + O(y^4). \tag{4.13}$$

A general argument for all m has been given by a thermodynamic Bethe ansatz method in [14].[10] We now describe how Δc is calculated using our approximation method,[11] and we will compare this with the exact result $\Delta c_{\text{exact}} = c(m) - c(m-1)$.

To construct the term $\langle T(z,\bar{z})T(0,0)\rangle$ we use the Ward identities which follow from euclidean invariance

$$\begin{aligned}\partial_{\bar{z}}\langle T(z,\bar{z})\Phi_1(x_1)\cdots\Phi_n(x_n)\rangle + \frac{1}{4}\partial_z\langle\Theta(z,\bar{z})\Phi_1(x_1)\cdots\Phi_n(x_n)\rangle = \\ = \pi\sum_{i=1}^{n}(\delta(z-x_i)\partial_{x_i} - \partial_z\delta(z-x_i)h_i)\langle\Phi_1(x_1)\cdots\Phi_n(x_n)\rangle\,,\end{aligned} \tag{4.14}$$

here Φ_i are primary fields with conformal dimension h_i. For the correlator we are interested in contact terms vanish and we get

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\langle T(z_1,\bar{z}_1)T(z_2,\bar{z}_2)\rangle = \frac{1}{4^2}\partial_{z_1}\partial_{z_2}\langle\Theta(z_1,\bar{z}_1)\Theta(z_2,\bar{z}_2)\rangle. \tag{4.15}$$

Θ is the infinitesimal generator for scale transformation (hence its vanishing in the CFT), and in a renormalisable field theory Θ must belong to the space spanned by the relevant and marginal fields defining the perturbation away from criticality in (4.9)

$$\Theta(x) \equiv 2\pi\sum_{i=1}^{N}\beta^i(g)\Phi_i(x)\,, \tag{4.16}$$

where $\beta^i(g)$ is the beta-function given in terms of the renormalised coupling constants g [8]. (4.15) can thus be written

$$\partial_{\bar{z}_1}\partial_{\bar{z}_2}\langle T(z_1,\bar{z}_1)T(z_2,\bar{z}_2)\rangle = \frac{\pi^2}{4}\beta^i(g)\beta^j(g)\partial_{z_1}\partial_{z_2}\langle\Phi_i(z_1,\bar{z}_1)\Phi_j(z_2,\bar{z}_2)\rangle\,. \tag{4.17}$$

[10] In [15] the first fixed points $\mathcal{M}(m)$, $m = 3, 4, \ldots, 12$ were found numerically using the exact renormalisation group.

[11] Another way of calculating Δc follows from the proof of the c-theorem [4], where it is shown that $R^2\frac{dC}{dR^2} = -\frac{3}{4}R^2\langle T^\mu_\mu T^\mu_\mu\rangle$ leading to Cardy's sum rule [12] $\Delta c = -\frac{3}{4}\int_0^\infty d(R^2)R^2\langle T^\mu_\mu T^\mu_\mu\rangle$. The correlator is normally only known perturbatively so this gives a perturbative estimate of Δc.

The correlator $\langle \Phi_i \Phi_j \rangle$ can be calculated in perturbative conformal field theory using the operator product expansion in the ultra-violet conformal field theory. The bare correlator is in the lowest order in λ_0 given by [3][12]

$$\begin{aligned}\langle \phi(x)\phi(0)\rangle &= \frac{\langle \phi(x)\phi(0) e^{\lambda_0 \int d^2x' \phi(x')}\rangle_{\mathcal{M}(m)}}{\langle e^{\lambda_0 \int d^2x' \phi(x')}\rangle_{\mathcal{M}(m)}} \\ &= \langle \phi(x)\phi(0)\rangle_{\mathcal{M}(m)} + \lambda_0 \int d^2x' \langle \phi(x)\phi(0)\phi(x')\rangle_{\mathcal{M}(m),\text{conn.}} + O(\lambda_0^2) \\ &= \frac{1}{|x|^{2(2-y)}}\left(1 + \lambda_0 \frac{4\pi b(y) A(y)}{y}|x|^y + O(\lambda_0^2)\right), \end{aligned} \tag{4.18}$$

where $\phi(x)$ is the bare field $\phi_{(1,3)}(x)$ and $A(y) = \frac{\Gamma(1-y)\Gamma(1+y/2)^2}{\Gamma(1-y/2)^2\Gamma(1+y)} = 1 + O(y^3)$. The operator product expansion coefficient $b(y)$ can be calculated from a Coulomb gas representation of the minimal models using the formulas in [13]

$$\begin{aligned} b(y)^2 = &\frac{16}{3}\frac{(1-y)^4}{(1-y/2)^2(1-3y/4)^2}\left(\frac{\Gamma(1+y/2)}{\Gamma(1-y/2)}\right)^4\left(\frac{\Gamma(1-y/4)}{\Gamma(1+y/4)}\right)^3 \times \\ &\times \left(\frac{\Gamma(1-y)}{\Gamma(1+y)}\right)^2\left(\frac{\Gamma(1+3y/4)}{\Gamma(1-3y/4)}\right) = \frac{16}{3} + O(y). \end{aligned} \tag{4.19}$$

Choosing the renormalisation conditions $\langle \phi(x,g)\phi(0,g)\rangle|_{|x|=\mu^{-1}} \equiv \mu^4$, the renormalised correlator and the β-function becomes [3]

$$\begin{aligned} \langle \phi(x,g)\phi(0,g)\rangle &= \frac{\mu^4}{|\mu x|^{2(2-y)}}\left(1 + \frac{4\pi A(y) b(y) g}{y}(|\mu x|^y - 1) + O(g^2)\right), \\ \beta(g) &= -yg - \pi b(y) g^2 A(y) + O(g^3), \end{aligned} \tag{4.20}$$

where $\phi(x,g)$ is the renormalised field and g is the renormalised coupling. The zeros of the $\beta-$function, the renormalisation group fixed points, are thus $g_{UV} = 0$, $g^*_{IR} = -y/\pi A(y) b(y)$ and therefore $g \in (-y/\pi A(y) b(y), 0)$ as the theory (4.9) lies between the two scaling limits. From the Callan–Symanzik equation we get the running coupling constant [3]

$$\bar{g}(|x|) = |\mu x|^y \frac{g}{1 - \frac{\pi A(y) b(y) g}{y}(|x\mu|^y - 1)}, \tag{4.21}$$

interpolating between g_{UV} for $|x| \to 0$ and g^*_{IR} for $|x| \to \infty$ and satisfying $\bar{g}(\mu^{-1}) = g$. Euclidean invariance allows us to write the correlator of the energy-momentum tensor as $\langle T(z,\bar{z})T(0,0)\rangle = \tilde{F}/(\tilde{R})2z^4$ in terms of the dimensionless quantity $\tilde{R} = \mu^2 z\bar{z}$. The differential equation (4.17) then becomes

$$\frac{\partial^2}{\partial \tilde{R}^2}\tilde{F}(\tilde{R}) = \frac{\pi^2\beta^2}{4\mu^4}\tilde{R}^2\frac{\partial^2}{\partial \tilde{R}^2}\langle \phi(\tilde{R})\phi(0)\rangle. \tag{4.22}$$

[12] conn. stands for the connected correlator.

A solution to this equation is given by[13]

$$\tilde{F}(\tilde{R}) = \frac{\pi^2\beta^2}{2\mu^4}\left(\tilde{R}^2\langle\phi\phi\rangle - 4\tilde{R}\int^{\tilde{R}} d\tilde{R}'\langle\phi\phi\rangle + 6\int^{\tilde{R}} d\tilde{R}'\int^{\tilde{R}'} d\tilde{R}''\langle\phi\phi\rangle\right) + \alpha_1 + \alpha_2\tilde{R}\,, \tag{4.23}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. The differential equation (4.22) is a boundary value problem as $\tilde{F}(\tilde{R})$ is known in the scaling limits

$$\tilde{F}(\tilde{R}) \to \begin{cases} c_{UV} & \text{for } \tilde{R} \to 0 \\ c^*_{IR} & \text{for } \tilde{R} \to \infty\,. \end{cases} \tag{4.24}$$

In the limit where $\tilde{R} \to 0$ the correlator $\langle\phi\phi\rangle$ scales as in $\mathcal{M}(m)$ i.e. $\langle\phi\phi\rangle \sim 1/\tilde{R}^{2h}$, hence $\tilde{R}^2\langle\phi\phi\rangle \sim \tilde{R}^y \to 0$, $\tilde{R}\int^{\tilde{R}} d\tilde{R}'\langle\phi\phi\rangle \sim \tilde{R}^y \to 0$, and finally $\int^{\tilde{R}} d\tilde{R}'\int^{\tilde{R}'} d\tilde{R}''\langle\phi\phi\rangle \sim \tilde{R}^y \to 0$ for $\tilde{R} \to 0$. This sets the boundary value $\alpha_1 = c_{UV}$. As $\tilde{F}(\tilde{R})$ attains a finite value for $\tilde{R} \to \infty$ then all linear terms in $F(\tilde{R})$ must cancel and we set $\alpha_2 = 0$. Integrating (4.20) and inserting the boundary conditions gives

$$\begin{aligned}\tilde{F}(\tilde{R}) = c_{UV} + \frac{\pi^2 g^2\tilde{R}^y}{2}\Bigg(&\frac{y(2-y)(3-y)}{y-1} + 2\pi A(y)b(y)g\Bigg(\frac{(2-y)(3-y)}{1-y} + \\ &+\tilde{R}^{y/2}\frac{(3y-4)(3y-6)}{3(\frac{3}{2}y-1)}\Bigg)\Bigg)\,.\end{aligned} \tag{4.25}$$

Determining the function $F(s)$ as $F(s) = \tilde{F}(\tilde{R})|_{\tilde{R}=1, g=\bar{g}(s)}$, the theory is fixed at the point of renormalisation $|x|^{-1} = \mu$, and all scale dependence is moved into the running coupling constant, which with $|x| \to s^{1/4y}$ becomes

$$\bar{g}(s) = \frac{g s^{1/4}}{1 - \frac{\pi A(y)b(y)g}{y}(s^{1/4}-1)}\,, \tag{4.26}$$

so that scale-transformations move around in the coupling constant space. The 1 loop renormalisation group improved approximation to $F(s)$ then becomes

$$\begin{aligned}F_1(s) = c_{UV} + \frac{\pi^2}{2}\bar{g}^2(s)\Bigg(&\frac{y(2-y)(3-y)}{y-1} + \\ &+2\pi A(y)b(y)\bar{g}(s)\left(\frac{(2-y)(3-y)}{1-y} + \frac{(3y-4)(3y-6)}{3(\frac{3}{2}y-1)}\right)\Bigg)\,.\end{aligned} \tag{4.27}$$

Here $k = 4$ as $F_1(s) = \tilde{F}_1(s^{1/4})$. To obtain the approximation for c_{IR} we then have to calculate the contour integral

$$I_1(\rho) = \frac{1}{2\pi i}\int_{C_0} ds\frac{e^{\rho/s}}{s}F_1(s). \tag{4.28}$$

[13] These equations directly generalises to the case with more couplings.

In the limit $m \to \infty$ the ultra-violet and infra-red fixed points are perturbatively close in coupling constant space as noted above, hence $F_1(s)$ correctly describes $F(s)$ in this limit and we should take $\rho \to \infty$ in (3.13) thus eliminating the contribution from the cut. The approximation then becomes $\lim_{\rho\to\infty} I_1(\rho) = \lim_{s\to\infty} F_1(s) = c^*_{IR}$ using (3.16), and the approximation in this limit thus equals the RG improved perturbative result which is

$$\begin{aligned}\Delta c_{pert} &= c_{UV} - c^*_{IR} = -\frac{\pi^2}{2}(g^*_{IR})^2 \left(\frac{y(2-y)(3-y)}{y-1} + \right. \\ &\quad \left. + 2\pi A(y) b(y) g^*_{IR} \left(\frac{(2-y)(3-y)}{1-y} + \frac{(3y-4)(3y-6)}{3(\frac{3}{2}y-1)} \right) \right) \\ &= \frac{3y^3}{16} + O(y^4)\,. \end{aligned} \tag{4.29}$$

This is equal to the asymptotic form of the exact value $c(m) - c(m-1)$ in (4.13). We wish to improve this result using the approximation (3.17). We rewrite the running coupling constant

$$\bar{g}(s) = \frac{g s^{1/4}}{1 - \frac{\pi A(y) b(y) g}{y}(s^{1/4} - 1)} = \frac{g^*_{IR} |\tilde{g}| s^{1/4}}{1 + |\tilde{g}| s^{1/4}}\,, \qquad \tilde{g} = \frac{g}{g - g^*_{IR}} \in (-\infty, 0)\,. \tag{4.30}$$

All dependence of the renormalised coupling g are now moved into the parameter ρ setting $s' = s|\tilde{g}|^4$ and $\rho' = \rho|\tilde{g}|^4$

$$I_1(\rho, g) = \frac{1}{2\pi i} \int_{C_0} ds \frac{e^{\rho/s}}{s} F_1(s) = \frac{1}{2\pi i} \int_{C_0'} ds' \frac{e^{\rho'/s'}}{s'} F_1(s') = \tilde{I}_1(\rho') \tag{4.31}$$

and then $\bar{g}(s') = \frac{g^*_{IR}}{1 + s'^{1/4}}$. This contour integral can be evaluated for example doing a numerical integration or a series expansion in $F_1(s')$, we have done both. A numerical integration of (4.31) with $m = 14$ using the NAG Fortran Library is shown in figure 5.

In figure 5 the dashed line indicates the expected behaviour of the exact function $\tilde{I}(0) - \tilde{I}(\rho')$ which is taken analogous to the curves of the free theories in figure 4. Writing $F_1(s')$ as a power series

$$F_1(s') = \Phi_1(\bar{g}(s')) = \sum_{n=0}^{\infty} h_n (s')^{n/4} \tag{4.32}$$

and inserting this into (4.31) and using the integral representation

$$\frac{1}{\Gamma(1+z)} = \frac{1}{2\pi i} \int_{C'_0} ds' \frac{e^{1/s'}}{s'} (s')^z\,, \tag{4.33}$$

we then obtain

$$\tilde{I}_1(\rho') = \sum_{n=0}^{\infty} \frac{h_n (\rho')^{n/4}}{\Gamma(1 + n/4)}\,, \tag{4.34}$$

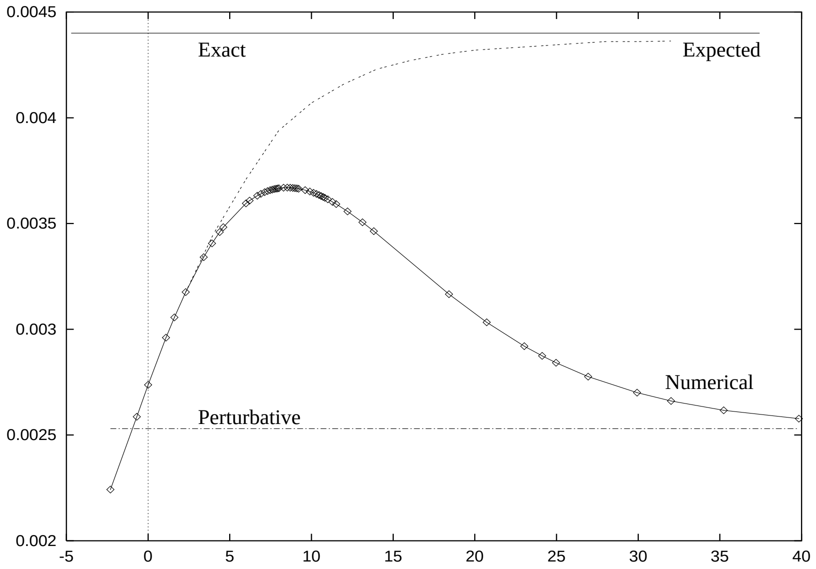

Figure 5: The numerical-result $c_{UV} - \tilde{I}_1(\rho')$ against $\log \rho'$ for $m = 14$. Also plotted is the exact and perturbative values Δc_{exact} and Δc_{pert}. The dashed line is the expected behaviour of $\tilde{I}(0) - \tilde{I}(\rho')$.

which is recognised as the Borel transform of order $k = 4$. This expression can be computed numerically (using e.g. Maple) by truncating to a finite n, and the minimal value can be found.

In table 1 the obtained values of the numerical integration denoted Δc_{approx} are listed together with the exact results Δc_{exact} and the RG improved perturbative values Δc_{pert}. For $m < 11$ the perturbative result will break down as Δc_{pert} becomes negative and thereby violates unitarity. In [1] we used $k = 2/y = (m+1)/2$, but here we have chosen $k = 4$ as we then get a stricter bound on the exact function as explained in section 3.3. The results for $k = 2/y$ and $k = 4$ are similar for all m calculated, except $m = 11, 12$ where they differ slightly. The improvement of the approximation (3.17) over the RG improved perturbative result is seen to be significant. In figure 6 $\Delta c_{\text{exact}} - \Delta c_{approx}$ and $\Delta c_{\text{exact}} - \Delta c_{pert}$ are plotted (scaled with $m(m^2-1)$ so that all the points can be distinguished) against m. The horizontal axis is then the exact value. The figure shows that the approximation improves the perturbative results with more than a factor two.

If we choose the maximum value $k = 1$ we get results which are almost identical to the exact values, we denote these by $\Delta c_{approx2}$, and plot in figure 7 Δc_{exact} and $\Delta c_{approx2}$ against m, the numbers are also listed in the table. In this case though we do not have a strict bound on the exact function $I(\rho)$ as for the case with $k = 4$ described in section 3.3. The exact function might still have a very small undershoot if for example the smallest non zero mass of a one particle state is small compared with ρ'_m, i.e. $\Delta c_{est} \sim \Delta c_{\text{exact}}$. Figure 7 shows the very good correspondence between Δc_{exact} and $\Delta c_{approx2}$, this is remarkable because the approximation is based on only a one loop calculation.

m	Δc_{exact}	Δc_{pert}	Δc_{approx}	$\Delta c_{approx2}$
11	0.00909	0	0.00642	0.00970
12	0.00699	0.00180	0.00533	0.00721
13	0.00549	0.00248	0.00437	0.00556
14	0.00440	0.00253	0.00368	0.00440
15	0.00357	0.00237	0.00310	0.00357
16	0.00294	0.00215	0.00262	0.00293
17	0.00245	0.00191	0.00222	0.00244
18	0.00206	0.00169	0.00190	0.00205
19	0.00175	0.00149	0.00163	0.00175
20	0.00150	0.00131	0.00140	0.00150
21	0.00130	0.00116	0.00122	0.00130
22	0.00113	0.00103	0.00109	0.00113
23	0.000988	0.000911	0.000956	0.000989
24	0.000870	0.000811	0.000846	0.000871
25	0.000769	0.000725	0.000752	0.000771
26	0.000684	0.000650	0.000671	0.000686
27	0.000611	0.000583	0.000601	0.000613

Table 1: The exact, perturbative and approximate values for Δc.

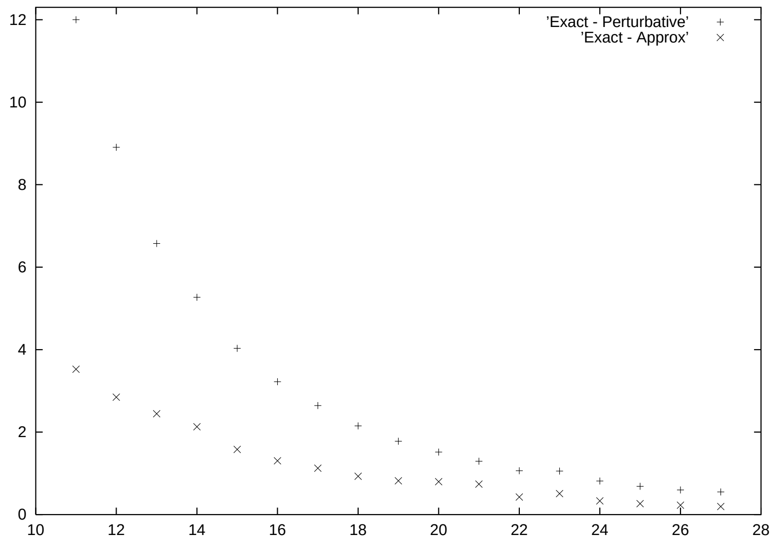

Figure 6: $(\Delta c_{\text{exact}} - \Delta c_{approx})m(m^2-1)$ and $(\Delta c_{\text{exact}} - \Delta c_{pert})m(m^2-1)$ against m.

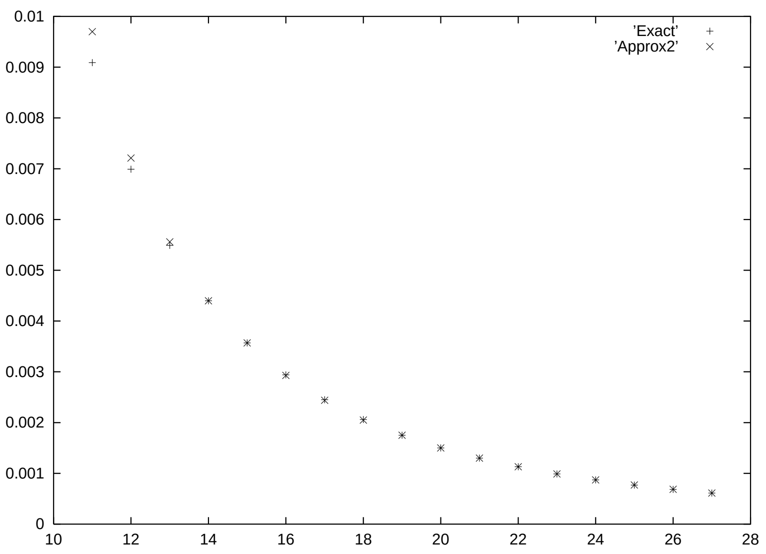

Figure 7: Δc_{exact} and $\Delta c_{approx2}$ against m.

5. Critical exponents for φ^4 in 3 dimensions

The other example we will consider is φ^4 theory in 3 dimensions with $O(N)$ symmetry.[14] We will study the Ising ($N = 1$) case in detail here, but the method applies for a general N. Quantum field theories have generic infra-red divergences in dimensions lower than 4, for φ^4 theory this can be seen by studying the 1PI 2 point function [17]. A way of regulating these infra-red divergences is to either do an ϵ expansion in $\epsilon = 4 - D$, or work with a massive theory. We will here do the latter, and work in $D = 3$ following [18]. We introduce the renormalised field, mass and coupling according to the conventions in [19] so the renormalised fields φ^R are given as: $\varphi = (Z_1)^{1/2}\varphi^R$, $\varphi^2 = Z_2(\varphi^2)^R$ and m, g denotes the renormalised mass and coupling. The infra-red divergences are now removed and only show up in the bare correlators as non analytic dependence in the bare coupling [18]. We will calculate two of the infra-red critical exponents ν and η, all other exponents follow from scaling relations,[15] e.g. Fisher's scaling relation $\gamma = \nu(2 - \eta)$.

[14] This model is one of the most studied models in critical phenomena and it is important because it shares its infra-red fixed point with a number of physical models, such as: polymers ($N = 0$), the Ising model ($N = 1$), super-fluid Bose-liquid ($N = 2$) and the Heisenberg ferromagnet ($N = 3$).

[15] The calculation of the critical exponents in φ^4 theory are among the most precise calculations in quantum field theory, and they have been calculated using both the ϵ expansion, exact renormalisation group arguments, perturbative quantum field theory, high temperature expansions, strong coupling expansions and monte Carlo simulations. These exponents describe the scaling behaviour of the theory in the scaling region and are universal in the sense that they are only determined by the infra-red fixed point and are therefore identical for all quantum field theories flowing to this point under the renormalisation group, irrespective of the underlying microscopic dynamics.

The free 2 point correlator, or propagator, is given as

$$G_0(p,m) = \int d^D x\ e^{ipx} \langle \varphi(x)\varphi(0)\rangle = \frac{1}{p^2+m^2}. \tag{5.1}$$

The mass is related to the critical temperature θ as $\theta \sim m^2$, so that the free propagator can be written as

$$G_0(p,\theta) = \theta^{-\gamma} h(\frac{p}{\theta^\nu}) \tag{5.2}$$

where h is regular at the origin and $\gamma = 1$ and $\nu = 1/2$. This scaling behaviour generalises to the interacting theory with critical exponents ν and γ, hence from the scaling relations all critical exponents can be obtained from the propagator. We will now define the function $\tilde{F}(s)$ above for respectively ν and η, and show that they are analytic in some sector with the correct scaling limits.

Using that the massless theory for small momentum (infra-red region) is equivalent to the large momentum region at $g = g_c$ [18] it follows from the Callan-Symanzik equation for the renormalised propagator that in the limit of small momentum it scales as $G(p) \sim 1/p^{2-\eta}$, hence defining

$$\tilde{F}_\eta(p) = p\frac{\partial G_2^R(p)/\partial p}{G_2^R(p)} + 2\,, \tag{5.3}$$

would satisfy the infra-red behaviour $\lim_{p\to 0} \tilde{F}_\eta(p) = \eta$. To get the ultra-violet behaviour we can look at the spectral representation

$$\tilde{F}_\eta(p) = -2\,\frac{\int_0^\infty d\mu^2\ \tilde{c}(\mu^2,m,g)\frac{p^2}{(p^2+\mu^2)^2}}{\int_0^\infty d\mu^2\ \tilde{c}(\mu^2,m,g)\frac{1}{p^2+\mu^2}} + 2 \to 0\,, \qquad \text{for } p\to\infty\,, \tag{5.4}$$

or simply use that $G_2^R(p) \to \frac{1}{p^2+m^2}$ in the ultra-violet (the spectral density will only contribute with $\delta(\mu^2-m^2)$), showing that $\lim_{s\to\infty}\tilde{F}_\eta(s) = 0$, as it should be using the scaling relation $\nu(2-\eta) = \gamma$. From this spectral decomposition it also directly follows that $\tilde{F}_\eta(s)$ is analytic for $s \in S(\pi-\epsilon')$ for $\epsilon' \ll 1$,[16] and that

$$\tilde{F}_\eta(s) \to \begin{cases} \eta & \text{for } s\to 0_+\,, \\ 0 & \text{for } s\to\infty\,. \end{cases} \tag{5.5}$$

Dimensional analysis gives us that

$$-m\frac{\partial G_2^R(p,m)}{\partial m} = p\frac{\partial G_2^R(p,m)}{\partial p} - [G_2]G_2^R(p,m) = \left(p\frac{\partial}{\partial p}+2\right)G_2^R(p,m) \tag{5.6}$$

[16] Actually it will be analytic in $S(\pi)$ but to get the Borel transform $\tilde{F}_\eta(s)$ needs to be bounded at the origin so that $\tilde{F}_\eta(s)$ have to be bounded for all closed subsets of S which would not be the case with $S(\pi)$.

hence it follows that

$$\tilde{F}_\eta(p) = -m\frac{\partial G_2^R(p,m)/\partial m}{G_2^R(p,m)} = Z_1^{-1} m\frac{\partial Z_1}{\partial m}\,, \tag{5.7}$$

also showing that the ultra-violet value is zero as the gaussian theory is ultra-violet finite. We also want to compute ν, it again follows from the Callan-Symanzik equation for the n point Green function with s insertions of φ^2 that the scaling behaviour for low momenta at the critical point is [19]

$$G_{n,s}^R(\lambda p, \lambda q, m) \sim \lambda^{D-\frac{1}{2}n(D+2-\eta)-s/\nu}. \tag{5.8}$$

Hence in the limit $p \to 0$

$$p\frac{\partial G_{2,1}^R(p,q,m)/\partial p}{G_{2,1}^R(p,q,m)} = -2+\eta-\frac{1}{\nu}\,, \tag{5.9}$$

so we define

$$\tilde{F}_\nu(p) = -\tilde{F}_\eta(p) + 4 + p\frac{\partial G_{2,1}^R(p,-p,q=0,m)/\partial p}{G_{2,1}^R(p,-p,q=0,m)}\,. \tag{5.10}$$

Using the same trick as before we see that $\tilde{F}_\nu(p)$ can be rewritten as

$$\tilde{F}_\nu(p) = -\frac{m\frac{\partial}{\partial m}(G_{2,1}^R(p,-p,q=0,m)/G_2^R(p,m)}{(G_{2,1}^R(p,-p,q=0,m)/G_2^R(p,m))} = {Z_2}^{-1} m\frac{\partial Z_2}{\partial m}\,, \tag{5.11}$$

which shows that $\lim_{p\to\infty}\tilde{F}_\nu(p) = 0$, again because Z_2 is a constant in this limit. The equation (5.10) shows that $\tilde{F}_\nu(s)$ is analytic in $S(\pi-\epsilon')$ for all $\epsilon' \in (0,\pi)$. This follows from writing $G_{2,1}^R = G_{2,1}Z_1^{-1}{Z_2}^{-1}$ and using that $G_{2,1}(p,-p,q=0) = -\frac{\partial}{\partial m_0^2}G_2(p)$ and then rewriting in terms of renormalised quantities and doing the spectral representation. $\tilde{F}_\nu(s)$ then satisfies

$$\tilde{F}_\nu(s) \to \begin{cases} 2-\dfrac{1}{\nu} & \text{for } s\to 0_+\,, \\ 0 & \text{for } s\to\infty\,. \end{cases} \tag{5.12}$$

The functions $\tilde{F}_\eta(g)$ and $\tilde{F}_\nu(g)$ have been calculated up to an amazing 7 loops in the coupling constant g. We will follow the convention of [21] where $\tilde{g} = \frac{3}{16\pi}g$ and $\tilde{\beta}(\tilde{g}) = \frac{3}{16\pi}\beta(g)$ giving ±1 as the first two coefficients in the $\tilde{\beta}$ function; we will use β, g to denote $\tilde{\beta}, \tilde{g}$ below to simplify the notation. The results are that [21]

$$\begin{aligned}
\tilde{F}_\eta(g) &= 0.0109739369g^2 + 0.0009142223g^3 + 0.0017962229g^4 - \\
&\quad - 0.0006536980g^5 + 0.0013878101g^6 - 0.001697694g^7 \\
\tilde{F}_\nu(g) &= \frac{1}{3}g - 0.0631001372g^2 + 0.0452244754g^3 - 0.0377233459g^4 + \\
&\quad + 0.0437466494g^5 - 0.0589756313g^6 + 0.09155179g^7 \\
\beta(g) &= -g + g^2 - 0.4224965707g^3 + 0.3510695978g^4 - \\
&\quad - 0.3765268283g^5 + 0.49554751g^6 - 0.74968893g^7\,,
\end{aligned} \tag{5.13}$$

where the beta-function has been calculated up to 6-loops [21]. The perturbative series for φ^4 theory was in [22] shown to be Borel summable in the coupling of order $k = 1$, so we set $F(s) = \tilde{F}(s^1)$. In [23] it was shown using the idea from [24] that the coefficients of the perturbative Green functions $f(g) = \sum_n f_n g^n$ for large n are given as

$$f_n = c(-\tilde{b})^n \Gamma(1 + b_l + n)(1 + O(1/n)) \,, \tag{5.14}$$

where $\tilde{b}, b_l, c$ were calculated in [23, 25]. $\tilde{b} = 0.14777422$ and it follows that the convergence radius of the Borel transform of $f(g)$ is given by $b_s = 1/\tilde{b}$ with a singularity at $-1/\tilde{b}$. When $b_l > 0$ we want to perform a Borel-Leroy transform of order k, which replaces a formal series $\hat{f} = \sum_n f_n z^n$ with $\sum_n (f_n/\Gamma(1+b_l+n/k) z^n$. Instead of (2.11) we then define

$$I(\rho) = \frac{\Gamma(1 + b_l)}{\rho^{b_l}} \frac{1}{2\pi i} \int_{\tilde{C}} \frac{ds}{s} s^{b_l} e^{\rho/s} F(s), \tag{5.15}$$

this function will then have the limits $\lim_{\rho\to\infty} I(\rho) = F_{IR}$ and $\lim_{\rho\to 0} I(\rho) = F_{UV}$ as before.

We introduce the scale parameter s via the exact running coupling $\bar{g}(s)$ ($s \to 0$ in the ultra-violet) then we know that $F_\eta(\bar{g}(s))$ and $F_\nu(\bar{g}(s))$ are asymptotic series in $\bar{g}(s)$. We now approximate the exact running coupling $\bar{g}(s)$ with the solution to

$$s\frac{d\bar{g}_n(s)}{ds} = -\beta_n(\bar{g}_n) \,, \tag{5.16}$$

with the boundary condition $\bar{g}'_n(0) = 1$, where $\beta_n(g)$ is the perturbative β function to the nth order. $F_\eta(\bar{g}_n(s))$, $F_\nu(\bar{g}_n(s))$ are then asymptotic series in s, which we will write as $F_j(s) = \sum_l (F_j)_l s^l$ for $j = \eta, \nu$. The Borel-Leroy transform given by (5.15) becomes

$$I(\rho, b_l, k) = \sum_{l=1}^{n+1} \frac{(F_j)_l \; \rho^l}{\Gamma(1 + b_l + l/k)} \,. \tag{5.17}$$

As we are only working with a truncated series we will set $k = 1$ (only infinitesimally different from $k = 1 + \delta$, $\delta \ll 1$), note we are summing up to $n + 1$ as the nth order β function has $n + 1$ terms, and F_j are known to the $(n + 1)$th order.

To find the critical exponents ν, η we will here do an analytical continuation in the Borel transform (which we do by a conformal mapping) followed by a Padé approximation. The details of this calculation are given in the appendix. We get the following estimates[17]

$$\nu = 0.625 \pm 0.004 \tag{5.18}$$

[17] The error in ν is seen to be larger than the one in η, in contrast to the errors form other methods. The reason for this is that we have chosen a conservative estimate where we have averaged over the results with to different β-functions (using its alternating behaviour) as explained in the appendix.

and

$$\eta = 0.0315 \pm 0.0020\,. \tag{5.19}$$

These numbers should be compared with $\nu = 0.6304 \pm 0.0013$ [20], $\nu = 0.6300 \pm 0.0015$ [26], $\nu = 0.6290 \pm 0.0025$ (ϵ expansion [20]), $\nu = 0.6289 \pm 0.0008$ (Monte Carlo) and for the other exponent $\eta = 0.0355 \pm 0.0025$ [20], $\eta = 0.032 \pm 0.003$ [26], $\eta = 0.0360 \pm 0.0050$ (ϵ expansion [20]), $\eta = 0.0374 \pm 0.0014$ (Monte Carlo), $\eta = 0.0347 \pm 0.001$ (strong coupling [27]), most of these numbers are taken from [20].

In the usual evaluation one determines the value of the critical coupling g_c and then evaluates the re-summed series at this point. The values then becomes very dependent on the estimate of g_c. One advantage of our method is that we do not have to estimate g_c, likewise we do not have to perform the Laplace integral, but instead the critical point is reached taking the limit in the Borel transform. The disadvantage of this method is that we do not have specific knowledge of the quantities r_c, b_l governing the asymptotic behaviour for the transformed series in the scale parameter. In the usual case of the perturbative expansion in the coupling these parameters are obtained from estimates of the higher order behaviour of perturbation theory.

6. Conclusion

We have described a general method of obtaining the infra-red limit of a physical quantity as an integral in the ultra-violet region. This was done using the analytic structure of Green functions in a complex scale parameter and by moving all scale dependence into the running coupling. The infra-red limit is then given as the limiting value of the Borel transform in the scale of the physical quantity. Changing the way in which the scale is introduced amounts to changing the order in the Borel transform.

We have tested this on two examples where the infra-red limit is well known, namely for the central charge of the perturbed unitary minimal models and the critical exponents of φ^4 theory in three dimensions. For the perturbed minimal models we showed how an approximation method can be obtained for calculating the central charge, and we get approximations close to the exact values already at one loop by using the largest domain of analyticity implied by spectral decomposition. For the φ^4 theory our estimates of the critical exponents are within the errors of other, more elaborate approximations.

The two β-functions differ only in the 7th order term, and here the F_η term is about a factor 100 smaller than the F_ν term resulting in less sensitivity to this averaging. The errors obtained could be lowered e.g. using one of the techniques from [26] or as in [29] using the information about the pole at r_c to obtain an extra order in the Padé table, but the aim here has not been to get a very low error, but to apply the method described above to a well known example.

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A. Padé approximation

The Borel transform is analytic in a sector $S(\epsilon'')$ where $\epsilon'' \ll 1$ [16]. The series given by (5.17) will have a pole at the negative real axis for some value $\rho = -r_c$ ($r_c > 0$) determining its convergence radius. We will now analytically continue $I(\rho)$ given by (5.17) doing the conformal transformation (as in [17])

$$t = \frac{\rho}{r_c + \rho}\,, \qquad \rho \notin (-\infty, -r_c]\,, \qquad \rho = \frac{r_c t}{1-t}\,, \tag{A.1}$$

so that $\tilde{I}(t, b_l) = I(\frac{r_c t}{1-t}, b_l)$. If we write (5.17) as $I(\rho) = \sum_l I_l \rho^l$ then $\tilde{I}(t)$ is given by the series

$$\tilde{I}(t) = \sum_{l=1}^{n+1} \tilde{I}_l\, t^l\,, \qquad \tilde{I}_l = \sum_{m=1}^{l} I_m r_c^m \binom{l-1}{m-1}\,. \tag{A.2}$$

If all poles of $I(\rho)$ lies in $(-\infty, -r_c]$, as argued in [26, 17], then $\tilde{I}(t)$ given by this series is convergent for $t \in [0, 1)$. We want to evaluate this expression at the convergence radius $t = 1$ (where $\tilde{I}(t)$ is regular), to do this we will use Padé approximants [28, 29]. In Padé approximation a function is approximated by a rational function, called $[L/M]$, with polynomials of degree L and M in the numerator and denominator (and the constant in the denominator is 1). If the n first terms of the function is known then these coefficients are matched with the coefficients of the polynomials where $0 < L + N \leq n$. One then forms the Padé table with entries $[L/M]$. The Padé approximation is based on the conjecture that there is a subsequence of diagonal Padé approximants $[L/L]$ which converge uniformly to the function, and this conjecture has shown to hold in practice. The diagonal Padé approximants are conformally invariant and are therefore independent of r_c above. Generally we should use Padé approximants close to the diagonal. Note that according to the Padé conjecture one should still expect convergence of the Padé approximants even if there are a finite number of poles within $|t| < 1$, which is the case if not all poles of $I(\rho)$ are in $(-\infty, -r_c]$.

We also have to determine the values of r_c and b_l. Let us first note that from the boundary condition $\bar{g}_n'(0) = 1$ we have that $\bar{g}_n(s) = s + O(s^2)$ and in $F_j(s) = \sum_l (F_j)_l s^l$ we thus have that $(F_j)_l = (\tilde{F}_j)_l + \cdots$ where $(\tilde{F}_j)_l$ is the lth coefficient of g in $F_j(g)$, this means that the convergence radius of the transformed series cannot be larger than the convergence radius of the series in the coupling g, i.e. $0 < r_c \leq b_s = 1/\tilde{b}$. Also we would suspect that $b_l \sim b'_l$, where b'_l is the Leroy parameter of the series in g. If b_l is chosen too large we will divide by more than the actual asymptotic increase in (A.2) and the approximation value will be to small, if b_l is chosen too small we expect to get a poor convergence in the Padé table. In the same way we expect the approximate value to be too small if r_c is chosen smaller than the actual convergence radius because the values we are calculating are increasing from zero. This is the behaviour we see in the tables and we estimate r_c and b_l to be respectively the largest and smallest value so that there is convergence in the Padé table.

From the asymptotic form of the β function (5.14) it follows that it is alternating, and an approximation to a truncated alternating asymptotic series $\sum_{l=1}^{n} f_l$ is to use the series $\sum_{l=1}^{n} \tilde{f}_l$ where $\tilde{f}_l = f_l$ for $l < n$ and $\tilde{f}_n = f_n/2$ [30]. We have also obtained the approximation where the perturbative β-function to the nth order (for $n = 5$ and $n = 6$) is approximated in this way.

Using the criteria mentioned above, for the choice of r_c, b_l we get the following Padé table[18]

$$\begin{pmatrix} \cdot & \cdot & 0.407663 & 0.405138 & 0.386709 & 0.396082 \\ \cdot & 0.362383^{*} & 0.405248 & 0.408042^{*} & 0.392824 & \\ 0.414164 & 0.395111 & 0.392686 & 0.395422 & & \\ 0.377688 & 0.392075 & 0.394118 & & & \\ 0.4843^{*} & 0.403122 & & & & \\ 0.406194 & & & & & \end{pmatrix} \tag{A.3}$$

for the function $F_\nu(s)$ leading to $F_{\nu,IR} = 0.402 \pm 0.007$, where the error is the intertabular error for the points chosen (which is here larger than the Baker-Hunter error [32]), and $r_c = b_s$, $b_l = 2.4$. Using the β function with $1/2$ times the last coefficient we get the Padé table (we get the best convergence for the 5. order expression)

$$\begin{pmatrix} \cdot & \cdot & 0.407663 & 0.405138 & 0.404239 \\ \cdot & 0.362383^{*} & 0.405248 & 0.403736 & \\ \cdot & 0.395111 & 0.404338 & & \\ \cdot & 0.400335 & & & \\ \cdot & & & & \end{pmatrix}, \tag{A.4}$$

giving $F_{\nu,IR} = 0.404 \pm 0.004$, again with $r_c = b_s$ and $b_l = 2.4$. Averaging over these two we get for the critical exponent $\nu = 0.626 \pm 0.003$. At this value of b_l and r_c we have the best convergence in the Padé table, a more conservative estimate of $F_{\nu,IR}$ is obtained by varying r_c and b_s in a region around these values and then sample the

[18] The numbers marked with a * has a pole close to or in the interval $(0, 1)$.

highest and lowest value for which there is some convergence in the Padé table.[19] This gives $F_{\nu,IR} = 0.40 \pm 0.01$ leading to

$$\nu = 0.625 \pm 0.004\,. \tag{A.5}$$

For the other exponent η we get approximately the same Padé tables using either the β-function as given in (5.13) or with a half times the last term, the convergent Padé table becomes

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0.0291884 & 0.0321865 & 0.0326601 & \\ \cdot & 0.0271003 & 0.0489937^{*} & 0.0327393 & & \\ 0.0323937 & 0.0310551 & 0.031359 & & & \\ 0.0321239 & 0.0312882 & & & & \\ 0.0312289 & & & & & \end{pmatrix}, \tag{A.6}$$

and we obtain the critical exponent $\eta = 0.0319 \pm 0.0010$, where $r_c = b_s$ and $b_l = 1.8$. Again varying around these values gives the estimate

$$\eta = 0.0315 \pm 0.0020\,. \tag{A.7}$$

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